RESEARCH STATEMENT

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1. Introduction

My research lies in the intersection of model theory and combinatorics, and is particularly concerned with how combinatorial properties of families of finite structures relate to properties of the automorphism group of corresponding infinite limit structures. A centerpiece of this field is the Fraïssé construction, which assigns a *homogeneous* limit structure, which has a very rich automorphism group, to a family of finite structures satisfying the *amalgamation property.*

Examples of homogeneous structures include the rational order $(\mathbb{Q}, <)$ (the Fraïssé limit of all finite linear orders) and the infinite random graph (the Fraïssé limit of all finite graphs). The Fraïssé construction and its generalization by Hrushovski have been used to produce structures with prescribed properties, providing solutions to major questions in model theory, as well as other fields including permutation groups, combinatorics, and topological dynamics.

One of the most studied combinatorial properties of a family of finite structures is the Ramsey property. Ramsey-type properties identify highly structured pieces in large, seemingly disordered structures. We postpone a formal definition.

The Ramsey property implies the amalgamation property, under a mild side condition known as the *joint embedding property* [31]. The following seminal result from [25] connects the Ramsey property for a family of finite structures to an unusually strong dynamical property of the automorphism group of its Fraïssé limit.

Theorem 1.1 (Kechris, Pestov, Todorčević)**.** *Let $G = Aut(M)$ for M homogeneous. Then G is* extremely amenable *iff the class of finite substructures of M has the* Ramsey property*.*

More generally, the Ramsey property, together with a related combinatorial property called the expansion property, allows for the computation of the universal minimal flow [3, 29, 32], an important dynamical invariant, for automorphism groups of countable structures.

Another example of a combinatorial property of families of finite structures is the *extension property for partial automorphisms*, which has several consequences for the automorphism group of the corresponding Fraïssé limit, including amenability [26].

The amalgamation and Ramsey properties also appear in more traditional combinatorial settings, and we list a few examples. Weakenings of these properties, namely 1-amalgamability and unsplittabiliy, have been applied to bound the growth of certain permutation patterns, as discussed in the survey [24]. Counting the orbits of the action of a homogeneous (or more generally ω-categorical) group yields familiar integer sequences. Finally, in constraint satisfaction problems, working over a homogeneous domain allows the machinery from finite domains to be carried over to infinite domains, and the Ramsey property has been a key tool in the further analysis of the complexity of such problems [4–6] .

2. Prior Work

2.1. **Homogeneous Permutation Structures.**

Definition 2.1. A structure M is *homogeneous* if any isomorphism between finite substructures of M extends to an automorphism of M.

Classifying all homogeneous structures of a given type is a common problem (e.g. [12, 13, 15, 27, 28]). For a homogeneous structure to be Ramsey, it must have a definable linear order [25], and so the classification of various homogeneous structures equipped with a linear order is of particular importance for identifying new examples in structural Ramsey theory [31].

This was the motivation for the classification of homogeneous ordered graphs. However, no new examples were uncovered by the classification, as it turned out every homogeneous ordered graph arises in simple fashion from an unordered homogeneous structure (either a graph, tournament, or partial order) [15]. To be more precise, these structures are interdefinable with generic expansions by a linear order of homogeneous graphs or tournaments, or generic linear extensions of homogeneous partial orders. It is natural to ask whether a similar statement might hold more generally, which would greatly simplify classifying homogeneous ordered structures.

Question 1. *Is every homogeneous ordered structure interdefinable with an expansion of a* homogeneous *proper reduct by a linear order?*

Is every ***primitive*** *homogeneous ordered structure interdefinable with an expansion of a* homogeneous *proper reduct by a* ***generic*** *linear order?*

The minimal case to test this would be to start with a structureless set, iteratively add linear orders, and observe the homogeneous structures that appear.

Definition 2.2. An n-dimensional permutation structure is a structure in a language consisting of n linear orders.

In the case $n = 2$, these structures may be viewed as permutations, with the first order providing the natural order and the second order providing the permuted order. Isomorphism types are then what combinatorialists call permutation patterns.

The classification of the homogeneous structures in the case $n = 1$ (linear orders) and $n = 2$ (permutations) were known, the latter carried out in [12], where the question of classifying the homogeneous n-dimensional permutation structures was originally posed. For $n = 1$, the only homogeneous structure is $\mathbb{Q}$, while for $n = 2$ the structures are interdefinable with ordered graphs, and so are part of the previous discussion.

My first and third papers [9, 11] are concerned with the classification of homogeneous n-dimensional permutation structures.

In [9], I undertook the first step in the classification of the homogeneous n-dimensional permutation structures: taking a "census" of naturally occurring examples. This led me to prove the following, via a construction producing many new imprimitive examples.

Theorem 2.3 ([9], B.)**.** *Let Λ be a finite distributive lattice. Then there is a homogeneous finite dimensional permutation structure whose lattice of $\emptyset$-definable equivalence relations is isomorphic to Λ.*

A subtle modification to the language used in the construction that yields Theorem 2.3 captures all known examples, and led to a conjectured classification of the homogeneous finite dimensional permutation structures, which, if true, falls into the regime suggested by Question 1. Further evidence in support of this conjectured classification was provided by another result from [9], suggesting that the lattice of $\emptyset$-definable equivalence relations in a homogeneous finite dimensional permutations ought to be distributive.

Theorem 2.4 ([9], B.)**.** *Let Λ be the lattice of $\emptyset$-definable equivalence relations in a homogeneous finite dimensional permutation structure $\mathcal{M}$. If the reduct of $\mathcal{M}$ to the language of equivalence relations from Λ is homogeneous, then Λ is distributive.*

In [11], I classified the homogeneous 3-dimensional permutation structures, and the conjectured classification is confirmed in this case.

Theorem 2.5 ([11], B.). *The homogeneous 3-dimensional permutation structures are as conjectured. In particular, they may be produced by the modified construction from Theorem 2.3.*

2.2. Structural Ramsey Theory.

Definition 2.6. Let $\mathcal{K}$ be a class of structures closed under isomorphism. Given $A, B \in \mathcal{K}$, let $\binom{B}{A}$ denote the set of substructures of B that are isomorphic to A. We will say $\mathcal{K}$ is a *Ramsey class* if for any $n \in \mathbb{N}$ and $A, B \in \mathcal{K}$, there is a $C \in \mathcal{K}$ such that if $\binom{C}{A}$ is colored with n colors, there is a $\widehat{B} \in \binom{C}{B}$ such that $\binom{\widehat{B}}{A}$ is monochromatic.

One of the main questions of the subject is whether every relational amalgamation class can be expanded to a Ramsey class by adding finitely many relations. (An example of Hrushovski's generalization of the Fraïssé construction disproved the stronger question replacing "homogeneous" with "ω-categorical" [18].)

In [21], a powerful general framework for proving that a given class of structures is Ramsey was presented. In particular, this framework applies to structures whose Fraïssé limit has a non-trivial model-theoretic algebraic closure operation. This allowed the authors to prove several new classes are Ramsey, including classes with nested definable equivalence relations, which had been largely unmanageable with previous methods. However, most of the examples presented in that paper had a Fraïssé limit with a unary algebraic closure operation, meaning the closure of a set was determined by the closures of its elements, which significantly simplified applying the framework.

In [10], I used the framework from [21] to find Ramsey expansions of generic Λ-ultrametric spaces, resulting in some of the first known Ramsey classes whose Fraïssé limits have a non-unary algebraic closure operation (other such classes were presented in [22]). Λ-ultrametric spaces, which arose in the construction for Theorem 2.3, are a convenient language for expressing structures equipped with a family of equivalence relations, and the proof of the Ramsey property for this class provides a blueprint for handling other classes with non-nested equivalence relations. The main theorem of [10] identifies the minimal Ramsey expansion for these structures, i.e. the relations that must be added to make the class Ramsey, and provides a restatement in terms of topological dynamics.

Theorem 2.7 ([10], B.). *Let Λ be a finite distributive lattice and Γ be the generic Λ-ultrametric space. For every meet-irreducible $E \in \Lambda$, expand Γ by a generic subquotient order from E to its successor, let $\vec{\Gamma}^{min} = (\Gamma, (<_{E_i})_{i=1}^n)$ be structure thus obtained, and $\vec{\mathcal{A}}_\Lambda^{\min}$ its finite substructures. Then*

(1) $\vec{\mathcal{A}}_\Lambda^{\min}$ is a Ramsey class and has the expansion property relative to $\mathcal{A}_\Lambda$.
(2) The logic action of $Aut(\Gamma)$ on $\overline{Aut(\Gamma) \cdot (<_{E_i})_{i=1}^n}$ is the universal minimal flow of $Aut(\Gamma)$.

3. Future Work

3.1. Decidability of Joint Embedding.
Given a countable hereditary class of finite structures, there exists a countable structure whose finite substructures are the given class, thus providing some notion of a limit structure, exactly when the class satisfies the *joint embedding property* (JEP), i.e. for any two structures in the class there exists another they embed into.

The JEP has been studied in the context of permutation pattern classes (i.e. classes of 2-dimensional permutation structures specified by forbidden substructures), under the name atomicity, and been used to bound growth rates and provide structure theorems. In [30], it was shown that whether a class is *natural*, a more restrictive requirement, is decidable from the forbidden patterns. However, the corresponding question for the JEP, posed in initially in 2005 [33] and asked again in 2017 [24], is open.

Question 2. *Given a finite set of forbidden permutations, is it decidable whether the corresponding permutation pattern class has the JEP?*

I have investigated the corresponding question for graphs via a reduction to the tiling problem, anticipating the argument used there will carry over to the permutation pattern case through the use of suitable coding techniques afforded by the existence of infinite antichains of permutations. For graphs, I have proven the following.

Theorem 3.1 (B.). *Given a finite set of forbidden induced subgraphs, it is undecidable whether the corresponding hereditary graph class has the JEP.*

3.2. **Model Theory of Permutation Pattern Classes.** The JEP provides a countable permutation embedding all *finite* permutations in a permutation pattern class. However, upon taking the permutations-as-structures viewpoint introduced in [12], a more incisive model-theoretic question is to ask for a countable permutation embedding all *countable* permutations in a permutation pattern class. The existence of such a countable universal structure can be viewed as a strengthening of the JEP or a weakening of amalgamation.

The universality problem has been extensively studied in the case of graphs [14]. However, the general theory developed there based on an algebraic closure operation depends on forbidding subgraphs rather than induced subgraphs. As induced substructures are the relevant concept for permutation pattern classes, new techniques will be necessary. In the graph case, the universal structures that arise are generally canonical, in the sense of being the unique countable model of the model companion of the theory of the hereditary family; we may hope for the same to be true for permutations.

Question 3. *Which classes of countable permutations, specified by avoiding finitely many patterns, contain a universal permutation?*

When is the above universal permutation canonical, i.e. when does the theory of a permutation pattern class have a model companion with a unique countable model?

Are these questions decidable given the forbidden patterns?

The question of the decidability of the JEP for permutation pattern classes may be a preparatory step toward the above questions; while the JEP is certainly necessary for the existence of a countable universal permutation, it has also been conjectured to be sufficient [23].

The existence of such a structure is an indication of model-theoretic tameness, and if a canonical limit theory is provided, further model-theoretic properties can be investigated. These properties often provide a means for separating highly structured classes from wild ones.

Question 4. *Does the existence of a canonical universal permutation (in the above sense), or further model-theoretic properties of that structure's theory (e.g. NIP) have combinatorial consequences for the corresponding permutation pattern class (e.g. bounds on growth rate)?*

Are these further model-theoretic properties decidable given the forbidden patterns?

See [35] for the use of an NIP-like tameness condition to bound the growth rate of hereditary families.

3.3. **Extension Property for Partial Automorphisms.** A theorem useful for proving the extension property for partial automorphisms (EPPA) for classes whose Fraïssé limit has a unary algebraic closure operation was recently proven in [19]. However, as was noted there, the EPPA is not known in any non-trivial cases with a non-unary closure. As evidenced in recent work such as [1], techniques used to establish the Ramsey property, centering around a method for completing partial structures, may also lead to the EPPA. As such a completion method was used to prove the Ramsey property for Λ-ultrametric spaces, they are a promising candidate for the EPPA.

Question 5. *Given a finite distributive lattice Λ, does the class of all finite Λ-ultrametric space have the EPPA?*

Hubička and Nešetřil have recently begun to consider a generalization of metric spaces [20], hereafter *generalized* Λ^**-metric spaces*, subsuming both Λ-ultrametric spaces and the generalized metric spaces introduced by Conant in [17]. The distances in a generalized Λ^*-metric space come from a commutative partially ordered monoid; Conant had previously considered the linearly ordered case. This provides a unified method for treating many homogeneous structures, such as Cherlin's catalog of metrically homogeneous graphs [15], as metric structures, with forbidden substructures arising from violations of the triangle inequality. As Conant has already proven the EPPA for his structures when the monoid is semi-archimedean [16], generalized Λ^*-metric spaces are a natural next step after Λ-ultrametric spaces.

Question 6. *For which partially-ordered, commutative monoids does the corresponding class of finite generalized* Λ^**-metric spaces have the EPPA?*

3.4. **Homogeneous Permutation Structures.** Several natural questions remain regarding the work from §2.1. During the classification of the 3-dimensional case, an "exotic" structure arose and was ultimately proven not to exist, but heavily using $n = 3$.

Question 7. *Do similar "exotic" structures arise for higher* n*?*

The construction behind the conjectured classification produces ordered Λ-ultrametric spaces, which are made interdefinable with permutation structures by adding further orders. The next question asks how efficiently this translation can be done, and is needed to determine what the conjecture actually says for a particular n.

Question 8. *Given a lattice* Λ*, what is the minimal* n *such that* Λ *is the lattice of* $\emptyset$*-definable equivalence relations in some homogeneous* n*-dimensional permutation structure?*

Perhaps the most obvious question is the following.

Question 9. *What are the homogeneous 4-dimensional permutation structures?*

This is of interest for several reasons. First, it may answer Question 7. Second, classifying the primitive cases will almost certainly require computer assistance, and the techniques developed ought to be helpful for other classification problems. Finally, this is the first case in which non-linear lattices of $\emptyset$-definable equivalences will appear, raising issues not considered in [11].

It is easy to see that homogeneous permutation structures have NIP, a model-theoretic tameness property. There are various notions of rank for NIP theories, including VC-density and dp-rank, and many open questions concerning the relations of these rank notions, so computing them for particular theories may be illuminating.

Question 10. *What are the dp-rank and VC-density of the theories of homogeneous permutation structures?*

3.5. **The Limit Theory of the Triangle-Free Process.** The triangle-free process is a graph process which starts with n vertices and adds edges randomly so long as they do not form a triangle. This process gives the best known lower bounds for the triangle Ramsey numbers $R(3, t)$, and its generalizations give the best known lower bounds for certain cases of the extremal Zarankiewicz problem [7, 8].

The limit theories of the random graph $G_{n,p}$ for various p have been intensively studied [34], particularly concerning 0-1 laws, i.e. whether every first order sentence is almost surely true or almost surely false as $n \to \infty$. For graph theorists, this answered questions about which values of p can be threshold functions, and for model theorists turned out to be intimately related to Hrushovski constructions [2].

Question 11. *Does the triangle-free process have a 0-1 law?*

If so, what are the model-theoretic properties of its limit theory?

If not, is the limit theory wild, e.g. does it interpret a suitably large fragment of arithmetic?

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